The Theory of CES Production Functions

Overview

- The Production Function
- Elasticity of Substitution
- The CES Production Function

The Production Function

- **A production function** defines the relationship between inputs and the maximum amount that can be produced within a given period of time with a given level of technology

$$\mathbf{Q = f(X_1, X_2, \ldots, X_k)}$$

Q = level of output

$X_1, X_2, \ldots, X_k$ = inputs used in production

The Production Function

For simplicity and graphical understanding, we will often consider a production function of two inputs:

$$\mathbf{Q = f(X_1, X_2)}$$

Q = output

X_1 = labor

X_2 = capital

Key assumption: whatever input or input combinations are included in a particular function, the output resulting from their utilization is at the maximum level

The Production Function

- **Short-run production function** shows the maximum quantity of output that can be produced by a set of inputs, assuming the amount of at least one of the inputs used remains unchanged

- **Long-run production function** shows the maximum quantity of output that can be produced by a set of inputs, assuming the firm is free to vary the amount of all the inputs being used

The Production Function

The production function is

○ a mathematical function that specifies the output of a firm, an industry, or an entire economy for inputs of labor, capital, energy etc.

○ an assumed technological relationship, based on the current state of engineering knowledge (it does not represent the result of economic choices)

○ a function that encompasses a maximum output for a specified set of inputs

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The Production Function

Production is the process of transforming inputs into outputs. The fundamental reality which firms must contend with in this process is ***technological feasibility***. The state of technology determines and restricts what is possible in combining inputs to produce output.

The most general way is to cenceive of the firm as possessing a ***production possibility set*** Y, where simply

$Y=\{x \epsilon R^m \mid x \text{ is a feasible production plan } \}$

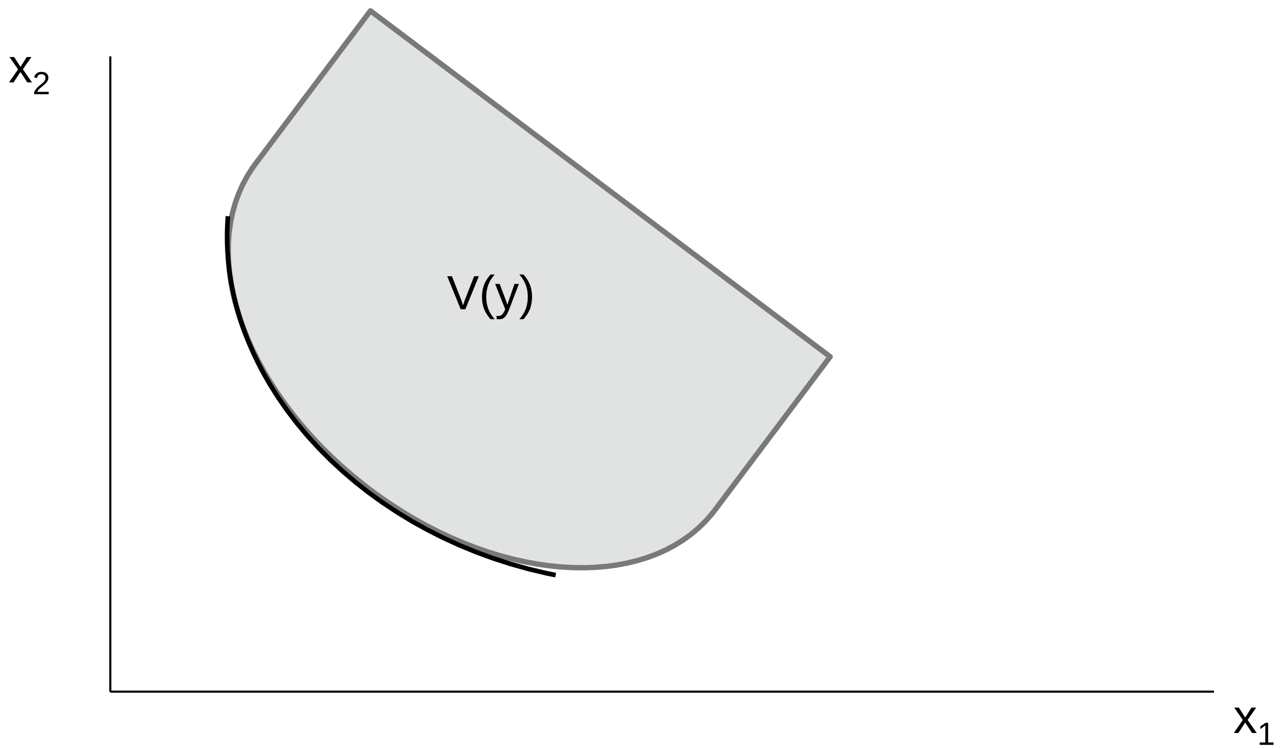
The Production Function

The production possibility set is a most general way to characterize a firm's technology by allowing for multiple inputs and multiple outputs. Most often, however, we will want to consider firms producing only a single product from many inputs. For that, it is more convenient to describe the firm's technology in terms of the inputs necessary to produce different amounts of the firm's output. This can best be done with the concept of the ***input requirement set***:

$$V(y)=\{x \mid x \in R^n,\ y \in R,\ (y,-\mathbf{x}) \in Y\}$$

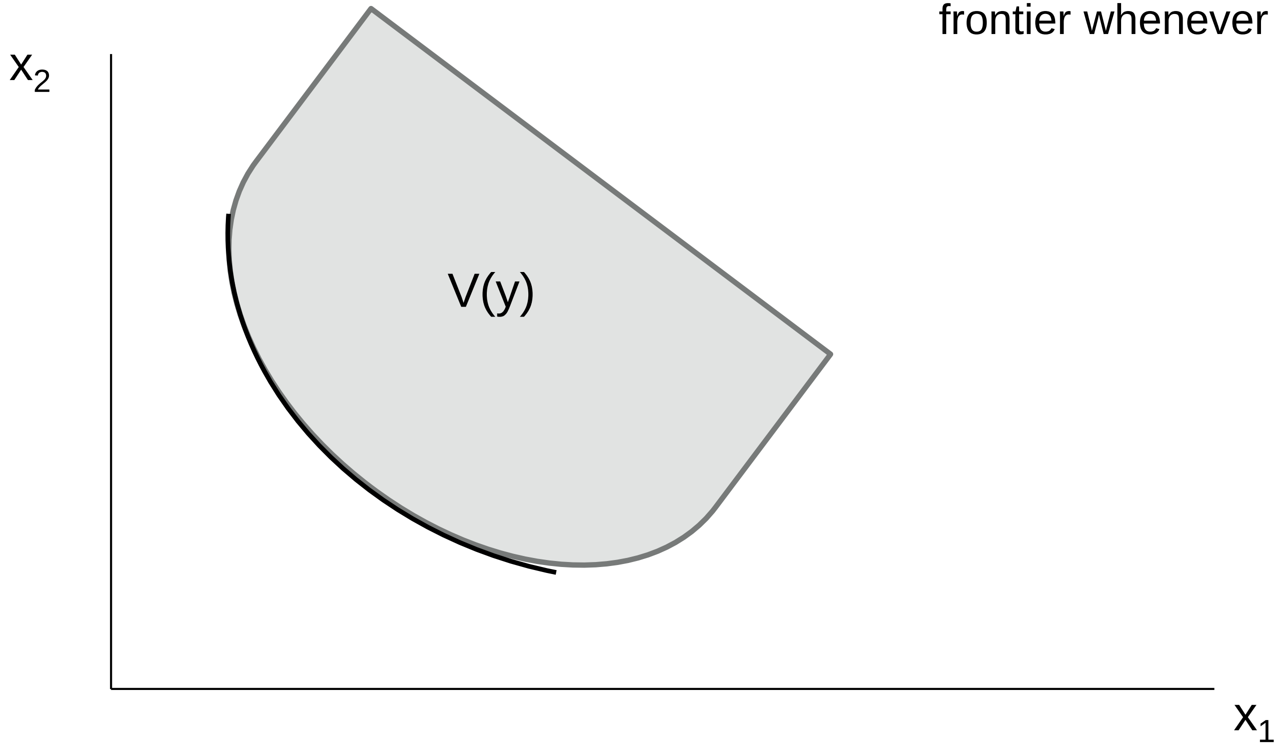
The Production Function

The ***input requirement set***: defines all combinations of inputs which produce an output level of at least y units.



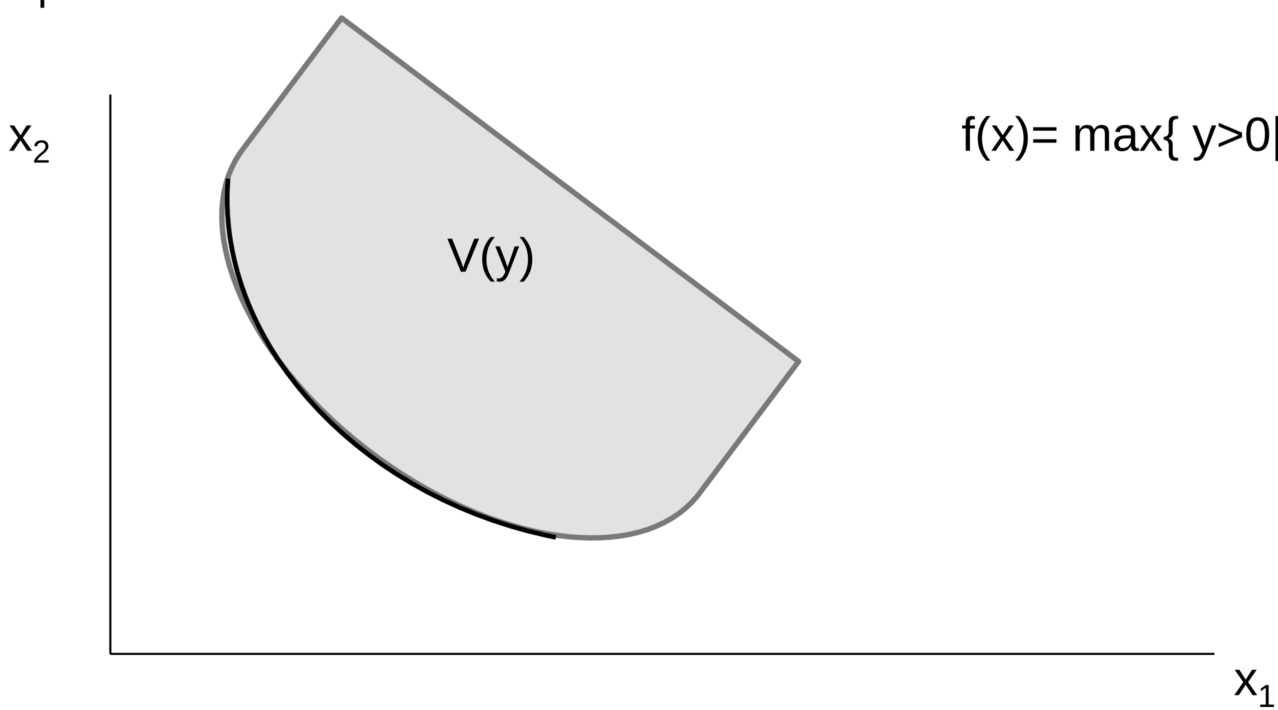
The Production Function

The boundary, called the isoquant, is important: it is the set of input vectors that produce exactly y units of output. The isoquant is the **efficient frontier** of the input requirement set. We would expect a firm producing y units of output to choose to operate at the efficient frontier whenever inputs are costly.



The Production Function

Yet another way to represent a firm's technology is a real valued ***production function***. The production function gives the maximum output that can be achieved from any vector of inputs, and so summarizes the efficient frontier of the production possibility set in the single-output case.



$f(x) = \max\{ y>0 | \mathbf{x} \in V(y) \}$

The Production Function

$f(\mathbf{x}) = \max\{\, y>0 \mid \mathbf{x} \in V(y) \,\}$

The partial derivative $\partial f(x)/\partial x_i$ is called the marginal product of factor i and gives the rate at which one factor can be substituted for another factor without changing the level of output produced.

The Producers' Problem

Max $f(\mathbf{x}) - \mathbf{p}_x \cdot \mathbf{x}$
Subject to **Tech. Constr.**

The Production Function

$y=f(\mathbf{x})= \max\{ y>0 | \mathbf{x} \in V(y) \}$

Marginal Product

- To study variation in a single input, we define Marginal Product as the additional output that can be produced by employing one more unit of that input while holding other inputs constant

$$\text{Marginal Product of } x_1 = MP_{x1} = \frac{\partial y}{\partial x_1}$$

$$\text{Marginal Product of } x_2 = MP_{x2} = \frac{\partial y}{\partial x_2}$$

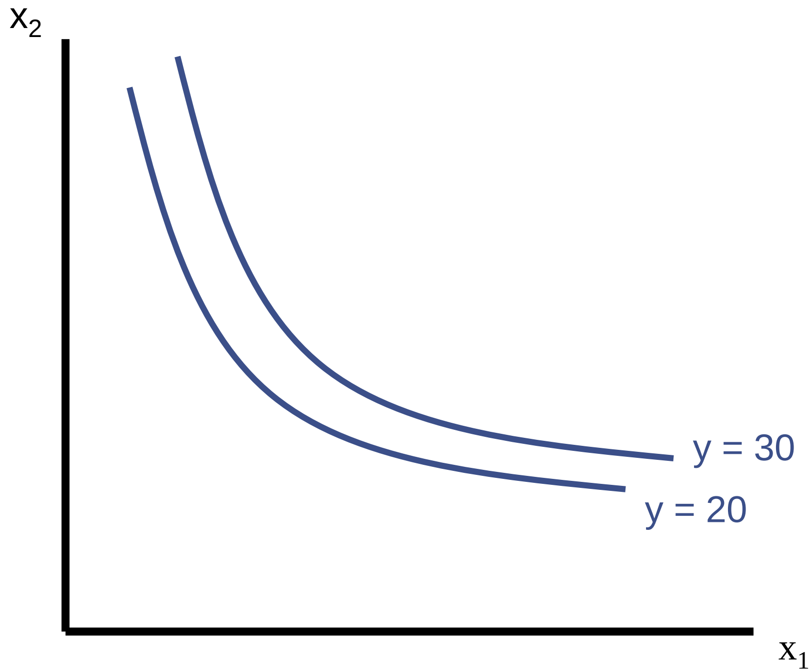
Isoquant Map

- To illustrate the possible substitution of one input for another, we use an isoquant map
- An isoquant shows those combinations of x_1 and x_2 that can produce a given level of output (y_0)

$$f(x_1,x_2) = y_0$$

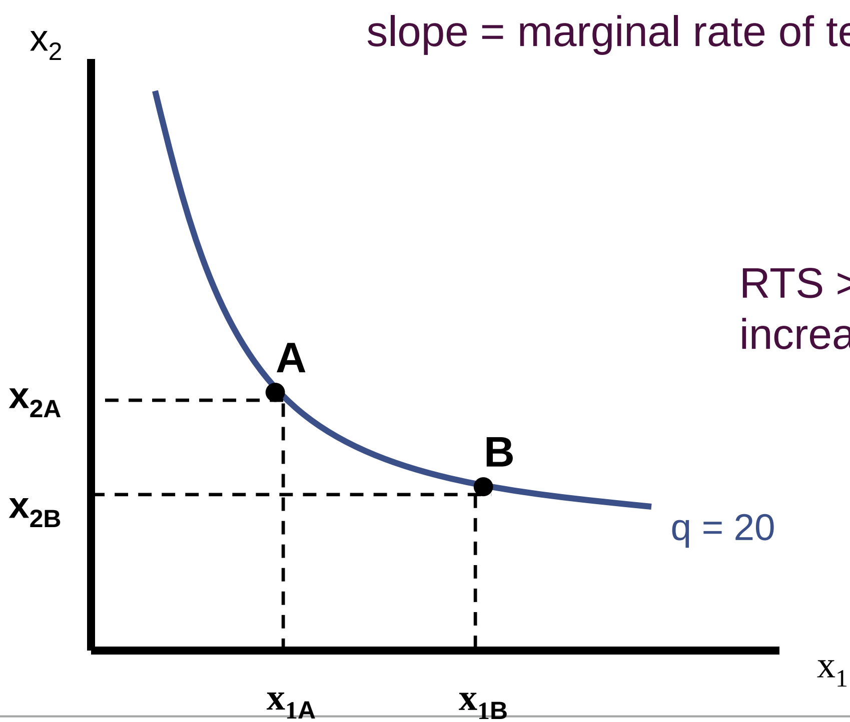
Isoquant Map

- Each isoquant represents a different level of output
  - output rises as we move northeast



Marginal Rate of Technical Substitution

- The slope of an isoquant shows the rate at which x_1 can be substituted for x_2

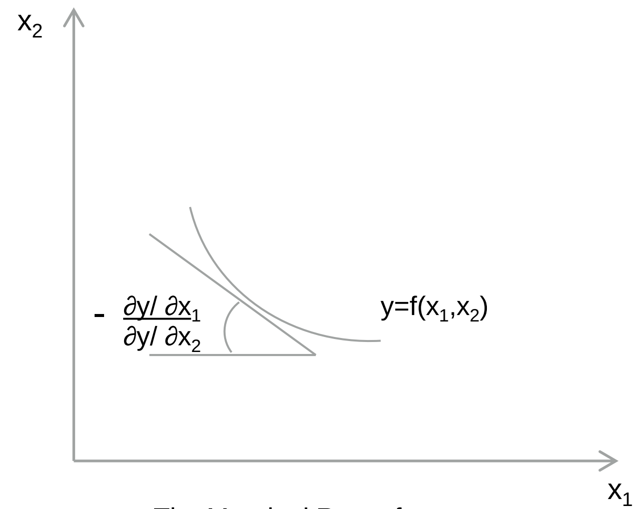


RTS > 0 and is diminishing for increasing inputs of x_1

Elasticity of Substitution

While the elasticity of a function of a single variable measures the percentage response of a dependent variable to a percentage change in the independent variable, the elasticity of substitution between two factor inputs measures the *percentage change in the factor proportions associated with a 1% change in the* ***MRTS*** *between them*.

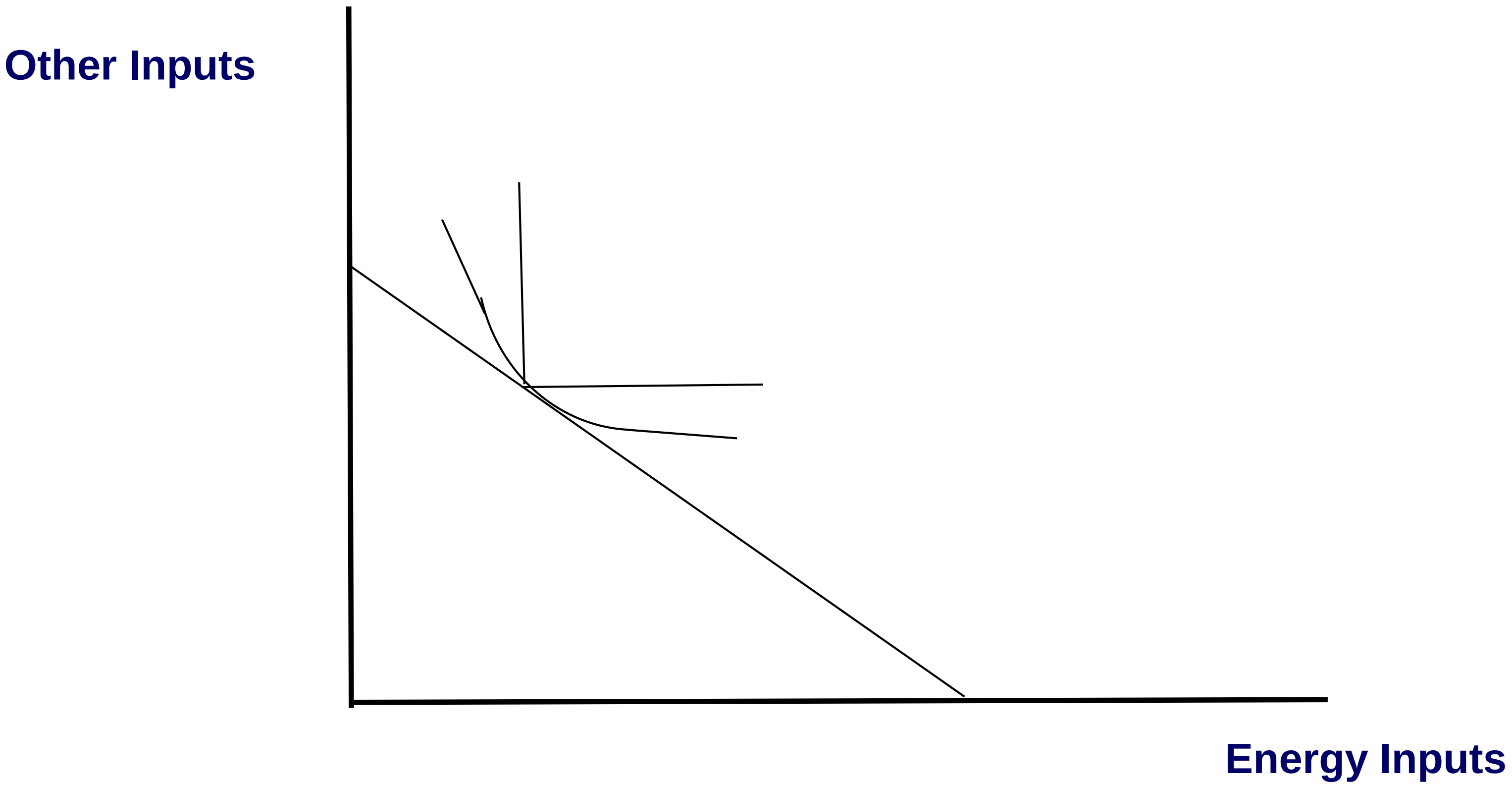
MRTS (Marginal Rate of Technical Substitution):
The rate at which one factor can be substituted for another without changing the level of output produced.



The Marginal Rate of Technical Substitution

The Theory of CES Production Functions

Elasticity of Substitution



$$\sigma = \frac{\text{\% change in quantity ratio of energy to other inputs}}{\text{\% change in price ratio of energy to other inputs}}$$

Elasticity of Substitution

Cobb-Douglas

$$Y = \prod_{i=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}}^{\alpha_{\mathrm{i}}}$$

Leontief

$$Y = \min\left\{x_1,\ldots, x_n\right\}$$

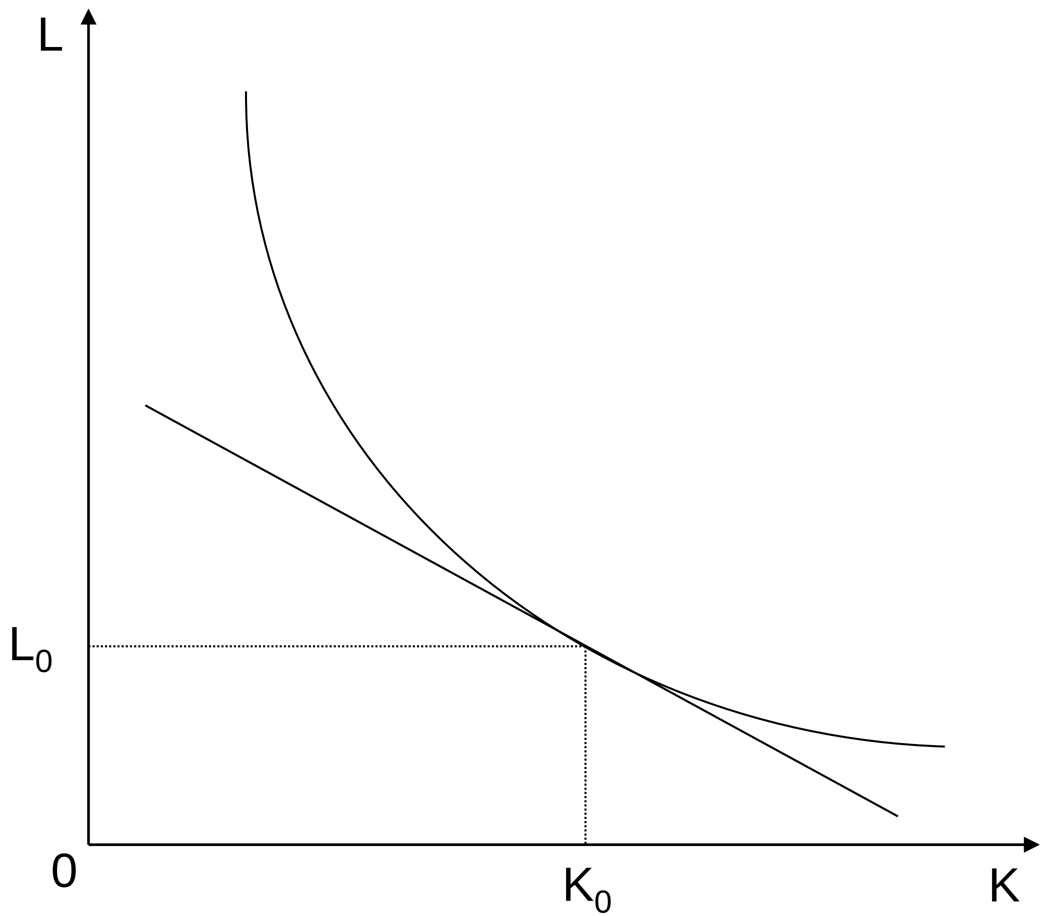
The CES Production Function

$$Y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho} \qquad \text{where} \quad \sum_{\mathrm{i}=1}^{\mathrm{n}} \alpha_i = 1$$

Q: What is the elasticity-of-substitution ?

The Production Function

CES Function Calibration



The Production Function

CES Function Calibration

Elasticities, benchmark quantities and prices determine the CES functions (technologies or preferences)

(i) benchmark demand quantities
$\rightarrow$ provide an anchor point for isoquants / indifference curves

(ii) benchmark relative prices
$\rightarrow$ fix the slope of the curve at that point

(iii) elasticity of substitution
$\rightarrow$ describes the curvature of the indifference curve